A COST-MINIMIZING ALGORITHM FOR SCHOOL CHOICE

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Abstract. The school choice problem concerns the design and implementation of matching mechanisms that produce school assignments for students within a given public school district. Previously considered criteria for evaluating proposed mechanisms such as stability, strategyproofness and Pareto efficiency do not always translate into desirable student assignments. In this note we propose methods to expand upon the notion of desirability for a given assignment mechanism by focusing on honoring student preferences. In particular we define two new student-optimal criteria that are not met by any previously employed mechanism in the school choice literature. We then use these criteria to adapt a well-known combinatorial optimization technique (Hungarian algorithm) to the school choice problem. In particular we create two mechanisms, each geared specifically to perform optimally with respect to one of the new criteria. Both mechanisms yield student-optimal outcomes. We discuss the practical implications and limitations of our approach at the end of the article.

1. Introduction

School choice policies are processes by which families have some say in determining where their children go to school. Since the late eighties such policies have been adopted by many school districts across the nation. Before school choice, students were typically assigned to public schools according to proximity. Since wealthy families have the means to move to areas with desirable or reputable schools, such families have always had *de facto* school choice. Children in families that could not afford such a privilege were left with no other option than to attend the closest school - whether or not the school was desirable and/or was a good fit. Thus school choice has been celebrated as a successful tool giving more families the power to shape their children's education, regardless of socioeconomic background.

In many school districts where funding and experienced teachers are lacking, school quality is uneven and often a small number of schools are strongly preferred over others. Since it is not possible to assign all students to their top choice school, the question of *how* to assign students to schools is often regarded as the central issue in school choice. In order to safeguard parents who seek to have their children attend schools conveniently within walking distance, at which a sibling is enrolled, or those offering need-based programs, districts define and adhere to a handful of school *priorities* which encapsulate such constraints. Thus school choice can be viewed as a two-sided matching problem. An extensive study of two-sided matching problems can be found in [26]; a more recent historical overview is [25].

Previous work on school choice as a matching problem evaluates assignments using the notions of stability, Pareto efficiency and strategyproofness. Though all worthy considerations, these do not necessarily suffice to promote the most desirable outcomes. In the context of school choice, stability corresponds to preventing priority violations. A priority violation occurs when a student desires a school more than the school to which she was assigned, and has higher priority than a student

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assigned to her desired school. Preventing priority violations is desirable for a very pragmatic reason: Students whose priorities are violated may have legitimate grounds for legal action. Even without legal recourse, it is often felt that students are "entitled" to schools in which they have been prioritized. However the focus on avoiding priority violations in current school choice mechanisms leads to documented inefficiencies. See [3], [9], [16], [24] for more on this potential tradeoff between stability and efficiency.

In this article, we propose an approach to the school choice problem which focuses on student preferences rather than school priorities. More specifically we introduce two new student-optimal measures of desirability for student-school matchings. The first is a "preference index" which reflects how closely student preferences are honored in an assignment. This index is consistent with previous notions of efficiency, since if one matching Pareto dominates another, it will also have a lower preference index. In addition, our index gives us the ability to compare two assignments that are Pareto incomparable. Our second measure looks at a collection of student preferences and makes precise the notion of its compatibility. Among other things this allows us to measure the compatibility of given preference profiles and to determine whether a particular assignment mechanism utilizes this compatibility to the fullest. We also show that no mechanism previously applied to the school choice problem is optimal with respect to our new criteria.

Our preference index naturally associates a "cost" to each matching and as a result we conceptualize the school choice problem as a "cost minimizing" assignment problem. In this context, a lower cost assignment corresponds to a matching that in some sense more closely meets student preferences. We can then introduce a new mechanism, adapted from a well-known combinatorial optimization algorithm, that produces a matching minimizing this cost, i.e., meeting student preferences as closely as possible. With respect to our index, the assignments produced by this mechanism often outperform the outcomes of standard mechanisms used or proposed in recent literature. Furthermore our method may be modified to respect and utilize student preference compatibilities (§§3.4). In cities without well-defined or legally required priorities (e.g. those that use whole-city lotteries), our approach provides policy makers two possible ways to create the most optimal student matching. Even cities committed to respecting student priorities may find these ideas valuable as priorities may indeed be incorporated at an intermediate or a final stage, see the relevant discussion in §4.

We summarize some relevant recent work on school choice in §§1.1. In §§1.2 we introduce the notation and standard terminology used throughout the rest of the paper and simultaneously describe our model. We define the preference index in §§2.1 and focus on our new measure of preference compatibility in §§2.2. We introduce our new mechanisms in §3, first providing a detailed description of the standard algorithm (§§3.1) and then explaining how we adapt it to the school choice problem (§§3.2, §§3.4). We study various properties of our mechanisms (§§3.3, §§3.4) and discuss some implementation issues (§§3.5). §4 concludes this article with a discussion of its implications and a view toward future work.

1.1. **Research background.** School district policy decisions have long provided active lines of inquiry for public policy designers, operations researchers, economists and education administrators. Much of the relevant work has focused on designing school district boundaries in order to optimize various measures. For a diverse yet representative selection of work in this vein, see [6], [7], [10], [11].

In our work we focus on assignment policy as a mechanism design problem, which provides a natural framework to investigate means of implementing social goals (cf. [19]). In the following we will refer to three specific mechanisms. The first two were introduced in [1] while the third was presented in [16].

(1) Student-Optimal Stable Matching Mechanism (SOSM)
(2) Top Trading Cycles Mechanism (TTC)
(3) Efficiency Adjusted Deferred Acceptance Mechanism (EADAM)

SOSM adapts the famous Gale-Shapley Deferred Acceptance (DA) algorithm [13] to the school choice problem. It is well established as a stable and strategyproof mechanism that has already been implemented in several large urban school districts [3], [5]. However, when applied to large-scale data SOSM may lead to some welfare losses [16]. TTC is an alternative mechanism which promotes efficiency as opposed to stability, and is also strategyproof. The basic algorithm is to create trading cycles alternating between students and schools and to allow efficient matchings. EADAM is proposed in [16] as a way to alleviate some of the efficiency costs of stability by iteratively running SOSM and modifying the preferences of any interrupters (who are students who cause others to be rejected from a school which later on rejects them) such that the SOSM outcome is Pareto dominated. As any Pareto domination of SOSM will lead to priority violations (cf. [13]), EADAM leads to at least one priority violation. We will not need the specific processes in our work.

Recent literature also examines various real-life mechanisms such as those from Boston [4], Chicago [8], Milwaukee [14], [27], and New York City [2].

1.2. **Notation, basic terms and our model.** Let I denote a nonempty set of students, and S a nonempty set of schools. A **matching** $M : I \to I \times S$ is a function that associates every student with exactly one school, or potentially no school at all. Write $\mathfrak{M}$ for the set of matchings. For convenience, let 0_s denote the **empty school** where $M(i) = (i, 0_s)$ implies that student i remains unmatched under M. We write $M_i = s$ if $M(i) = (i, s)$.

For all $s \in S$, let q_s denote the **capacity** of s, and define the **roster** of s under M as

$$\Lambda_s(M) = \{i \in I : M(i) = (i, s)\}.$$

More precisely, the roster $\Lambda_s(M)$ of s is the set of students in I who are assigned to s under M. Certainly $|\Lambda_s(M)| \leq q_s$. When the inequality is strict, we say that s is **unfilled**.

A **preference profile** for a student $i \in I$, written P_i, is a tuple $(S_1, \dots, S_n)$ where the S_j's form a partition of S and every element of S_j is preferred to every element of S_k if and only if $j < k$. Define the **ranking function** $\varphi_i : S \to \mathbb{N}$ of a student $i \in I$ by letting $\varphi_i(s)$ denote i's ranking of $s \in S$. In other words $\varphi_i(s) = j$ if $s \in S_j$. When each S_j is singleton, we say that i's preference profile is **strict**, (in which case we can view P_i as an n-vector). If $s_k, s_l \in S_j$ for some $j, k \neq l$, then we say that the student is **indifferent** between s_k and s_l. If i prefers s_k to s_l, we write $s_k \succ_i s_l$, or simply $s_k \succ s_l$ if i is unambiguous. We denote a set consisting of preference profiles for each student in I by $\mathbf{P} = \{P_i : i \in I\}$ and the space of all such sets is denoted by $\mathfrak{P}$.

A **priority structure** for a school $s \in S$, written Π_s, is a tuple $(I_1, \dots, I_n)$ where the I_j's form a partition of I and every element of I_j is preferred to every element of I_k if and only if $j < k$. When each I_j is singleton, we say that s's priority structure is **strict**, (in which case we can view Π_s as an n-vector). If $i_k, i_l \in I_j$ for some $j, k \neq l$, then we say that the school is **indifferent** between i_k and i_l. If s prefers i_k to i_l we write $i_k \succ_s i_l$, or simply $i_k \succ i_l$ if s is unambiguous from context. We denote a set consisting of priority structures for each school in S by $\mathbf{\Pi} = \{\Pi_s : s \in S\}$ and the space of all such sets is denoted by $\mathfrak{k}$.

A matching M' **(Pareto) dominates** M if $M'_i \succ_i M_i$ for all i and $M'_j \succ_j M_j$ is strict for some j. A **(Pareto) efficient matching** is a matching that is not (Pareto) dominated.

A **matching mechanism** $\mathcal{M} : \mathfrak{P} \times \mathfrak{k} \to \mathfrak{M}$ is a function that takes an ordered pair $(\mathbf{P}, \mathbf{\Pi}) \in \mathfrak{P} \times \mathfrak{k}$ of preferences and priorities and produces a matching.

Let Π_s be a priority structure for school s. We say that a matching M **violates the priority** of $i \in I$ for s if there exist some $j \in I$ and $s' \in S$ such that

(1) $M_j = s$, $M_i = s'$: j gets assigned s under M and i gets assigned s' under M.
(2) $s \succ_i s'$: i prefers attending s over s'; and
(3) $i \succ_s j$: s prioritizes i over j.

We say that a matching M is **stable** if

(1) M does not violate any priorities.
(2) No student is matched to a lower-ranked school when a more preferred school is unfilled.

A **stable mechanism** is a matching mechanism that always produces stable matchings.

A matching mechanism is **strategyproof** if there is no rational incentive for a student to misrepresent their preferences.

In the above context, the goal of the school choice problem is to find a matching mechanism $\mathcal{M}$ which satisfies certain criteria. The standard criteria for evaluating the desirability of a given mechanism are stability, (Pareto) efficiency, and strategyproofness.

2. New criteria for evaluating assignment mechanisms

The current literature on school choice proposes mechanisms that balance student preferences and school priorities. In our work, we emphasize student preferences. With this spirit, we introduce two new student-optimal criteria to expand the scope of what constitutes a good mechanism.

The question of what criteria to use to judge the quality or desirability of a mechanism is a difficult one; for example, see [20] where McFadden argues that tolerance of behavioral faults should be included in such a list of criteria. The goal of school districts when designing a school choice policy is not singular (unlike, for instance, the case of auction design where our sole objective is to maximize selling price). Thus, it is especially important for us to define a yardstick by which we measure the success of a given school choice mechanism. We could technically define the best school mechanism as one that minimizes the government education funding budgets, produces the most elite students, or improves the conditions of less-advantaged students the most, etc. Obviously, the ultimate design depends on how we define the objective/criteria of the school choice problem.

In this section we propose two criteria which provide two different interpretations of how to best honor student preferences. Our index (§§2.1) inherently captures the utilitarian objective of improving the outcome for as many students as possible,[1,2] while our notion of rank compatibility (§§2.2) aims to uphold the Rawlsian difference principle[3] by never giving a student a lower assignment than it needs to. Note that in the following, we will not address specific utility concerns or issues about preference intensities. Instead we will focus exclusively on preference profiles and the ordinal rankings these signify, ignoring, in particular, the *utility gap*s between the rankings.[4] We

[1] Note that we will be focusing on satisfying preferences as our measure of utility, and thus our approach may be more akin to preference utilitarianism a la R. M. Hare (1919–2002) (cf. [15]) rather than a more traditional view a la J. Bentham or J. S. Mill.

[2] Admittedly there are fairness issues to consider when using any utilitarian approach. If we make a majority absolutely giddy with pleasure at the expense of a handful who are made truly miserable, does this mean that the school district officials assigned with the public duty of serving everyone have failed to perform their duties effectively? We will address similar fairness issues later in §4.

[3] John Rawls (1921-2002) proposed the difference principle as one of two foundational principles of his theory of justice. According to the difference principle, social and economic inequalities should be arranged so that "*they are to be of the greatest benefit to the least-advantaged members of society*" [23, p.301].

[4] We could alternatively just invoke the principle of insufficient reason and assume that they are uniform.

will also be rethinking the role of school priority profiles as they apply to the school choice problem as we devise various ways to incorporate them into our framework (cf. §§3.2; also see §4 for an extensive discussion of priorities).

2.1. **A preference reverence index.** We begin with an example from [24] which illustrates the possible tradeoff between stability and efficiency mentioned in §1. Assume there are three schools, s_1, s_2, s_3 and three students i_1, i_2, i_3. The priorities of the schools and the preferences of the students are given by:

$$\begin{array}{ll} s_1 : i_1 \succ i_3 \succ i_2 & \qquad i_1 : s_2 \succ s_1 \succ s_3 \\ s_2 : i_2 \succ i_1 \succ i_3 & \qquad i_2 : s_1 \succ s_2 \succ s_3 \\ s_3 : i_2 \succ i_1 \succ i_3 & \qquad i_3 : s_1 \succ s_2 \succ s_3 \end{array}$$

Here, the only stable matching is:

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}.$$

But this matching is (Pareto) dominated by:

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_1 & s_3 \end{pmatrix}.$$

We see that the second matching (Pareto) dominates the first matching because it gives i_1 and i_2 schools they preferred over their stable matching. However, this (Pareto) efficient matching is not stable because i_2 now violates i_3's priority in s_1.

Now, we construct a "preference reverence index" that, on the students' side, would quantify how much their preferences were dismissed, and a "priority reverence index" that, on the schools' side, would quantify how much their priorities were dismissed. We assign numbers to the preferences and priorities of schools. Namely, we assign a 0 to a student's first choice, a 1 to their second choice, and so on. For the stable matching we have:

Schools: 0 + 0 + 2 = 2 (priority reverence index)

Students: 1+ 1+ 2 = 4 (preference reverence index)

And, for the (Pareto) efficient matching we have:

Schools: 1 + 2 + 2 = 5 (priority reverence index)

Students: 0 + 0 + 2 = 2 (preference reverence index)

We see that the (Pareto) efficient matching has better served the student preferences (lower preference reverence index) at the expense of the school priorities (higher priority reverence index). In fact, by the definition of (Pareto) efficiency, all (Pareto) dominations of stable matchings under SOSM improve how well student preferences are honored, thereby lowering the preference index.

Let us make the above precise.

Let I be a nonempty set of students, and S be a nonempty set of m schools. Recall that for any $i \in I$, $s \in S$, $\varphi_i(s)$ is i's ranking of s and for any matching $M : I \to I \times S$, $M_i = s$ denotes that $M(i) = (i, s)$. Let $\mathfrak{M}$ be the set of matchings. Define $\mu : \mathfrak{M} \to \mathbb{N}$ by

$$\mu(M) = \sum_{i \in I} (\varphi_i(M_i) - 1).$$

For any given $M \in \mathfrak{M}$ we will call $\mu(M)$ the **preference reverence index** of M or simply the **preference index**.[5]

[5]Similarly we could make precise the notion of our "priority reverence index". We will not fully investigate that direction in this article. Nonetheless also see §4.

For a given set of schools and students each equipped with priorities and preferences respectively, if there exists a matching M such that $\mu(M) = 0$, we will say that the students are **preference compatible**. If there exists no such matching, the students are **preference incompatible**. In §§2.2 we will refine this notion further.

Since $\mathfrak{M}$ is finite, $\mu(\mathfrak{M})$ is finite and hence there exists some $M \in \mathfrak{M}$ such that $\mu(M) \leq \mu(M')$ for all $M' \in \mathfrak{M}$. We will describe a method of seeking and locating such a minimal index matching in §3. Here we focus on some properties of our new index.

First we investigate how our index relates to the standard notion of (Pareto) efficiency. Here is a useful lemma:

Lemma 2.1. *Let M, $M' : I \to I \times S$ be two matchings. If M (Pareto) dominates M', then $\mu(M) < \mu(M')$.*

Proof. Since M dominates M', there exists at least one student i who is better off under M than M'. In other words $\varphi_i(M_i) < \varphi_i(M'_i)$. Since for all $j \in I \backslash \{i\}$, we have $\varphi_j(M_j) \leq \varphi_j(M'_j)$, this yields $\mu(M) < \mu(M')$. □

It clearly follows that our index is consistent with the more standard notion of (Pareto) efficiency. However, the index can also distinguish between two Pareto incomparables, and thus provide a way to define different levels of efficiency. Consider for instance the following:

Example 2.2. Assume we have three students and three schools, and each school has one spot. The student preferences over the schools are given as:

$$\begin{aligned} i_1 &: s_1 \succ s_2 \succ s_3 \\ i_2 &: s_3 \succ s_2 \succ s_1 \\ i_3 &: s_3 \succ s_1 \succ s_2 \end{aligned}$$

Here are two possible Pareto efficient matchings:

$$\text{Pareto Efficient Matching \#1} \quad (\mu = 1): \qquad \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

$$\text{Pareto Efficient Matching \#2} \quad (\mu = 2): \qquad \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_3 & s_2 \end{pmatrix}$$

Both solutions are (Pareto) efficient because in each, there is no way to make any of the students better off without making another student worse off. However, they are (Pareto) incomparable as moving from one to the other would necessarily harm at least one student. When we compare the two in terms of their preference indices, we see that one has a lower index than the other, and hence in terms of the index, Matching 1 is preferable to Matching 2. In other words, all Pareto efficient matchings are *not* made equal, and the preference index provides a way in which we can differentiate between (Pareto) efficient matchings. We can thus have some basis by which to rate some "better" while rating others "worse" provided that we are only given a list of ordinal preferences.

Next we focus on our index and its relation to stability. We begin with a result that follows from the above Lemma:

Corollary. *Let $\mathcal{M}$ be a stable mechanism and let $\mathcal{DA}$ be the deferred acceptance algorithm. Then for any preference profile, $\mu(\mathcal{DA}(P)) \leq \mu(\mathcal{M}(P))$.*

Proof. Follows from Gale-Shapley result [13] that the Deferred Acceptance algorithm (Pareto) dominates all other stable matchings. □

There can exist two stable matchings with the same preference index. Here is an example:

$$\text{Student preferences :} \quad \begin{array}{l} i_1 : s_4 \succ s_2 \succ s_6 \succ s_3 \succ s_1 \succ s_5 \\ i_2 : s_5 \succ s_3 \succ s_6 \succ s_4 \succ s_2 \succ s_1 \\ i_3 : s_2 \succ s_1 \succ s_6 \succ s_3 \succ s_5 \succ s_4 \\ i_4 : s_6 \succ s_5 \succ s_3 \succ s_2 \succ s_4 \succ s_1 \\ i_5 : s_3 \succ s_5 \succ s_4 \succ s_1 \succ s_2 \succ s_6 \\ i_6 : s_1 \succ s_2 \succ s_6 \succ s_4 \succ s_5 \succ s_3 \end{array}$$

$$\text{School priorities :} \quad \begin{array}{l} s_1 : i_5 \succ i_2 \succ i_4 \succ i_1 \succ i_3 \succ i_6 \\ s_2 : i_6 \succ i_4 \succ i_5 \succ i_1 \succ i_2 \succ i_3 \\ s_3 : i_3 \succ i_1 \succ i_2 \succ i_6 \succ i_5 \succ i_4 \\ s_4 : i_2 \succ i_4 \succ i_5 \succ i_1 \succ i_3 \succ i_6 \\ s_5 : i_5 \succ i_4 \succ i_2 \succ i_1 \succ i_3 \succ i_6 \\ s_6 : i_1 \succ i_3 \succ i_5 \succ i_4 \succ i_6 \succ i_2 \end{array}$$

Here, we have two stable matchings with the same preference index:

$$\text{Stable Matching \#1} \quad (\mu = 2) : \quad \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\ s_4 & s_3 & s_2 & s_6 & s_5 & s_1 \end{pmatrix}$$

$$\text{Stable Matching \#2} \quad (\mu = 2) : \quad \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\ s_4 & s_5 & s_1 & s_6 & s_3 & s_2 \end{pmatrix}$$

However, there is only one lowest preference index stable matching:

Lemma 2.3. *The SOSM matching is the unique lowest preference index stable matching for a given preference profile. In other words the inequality in the above Corollary is strict.*

Proof. This also follows from Gale and Shapley [13]: SOSM Pareto dominates all other mechanisms which prevent priority violations and the SOSM solution is unique. □

The preference index measures how well ordinal preferences are being honored as a whole. Each time we move to the next-best choice in a student's ranking, this counts as "1 violation" of their preferences, and we then add up the number of times we make such violations. Thus, perhaps a more apt title would be "preference dismissal index" since it is a measure of how little the preferences are being "honored" or "revered." It should be noted that the preference index assumes that it is the same to give one student their fifth choice and one their first choice (Total=4) as it is to give two students their third choice (Total=4).

2.2. **Rank compatibility.** Consider the student preference profile below

$$\begin{array}{l} i_1 : s_4 \succ s_1 \succ s_3 \succ s_2 \succ s_1 \\ i_2 : s_3 \succ s_2 \succ s_1 \succ s_3 \succ s_4 \\ i_3 : s_3 \succ s_1 \succ s_5 \succ s_4 \succ s_2 \\ i_4 : s_1 \succ s_4 \succ s_3 \succ s_5 \succ s_2 \\ i_5 : s_1 \succ s_4 \succ s_5 \succ s_3 \succ s_2 \end{array}$$

Here is a lowest preference index matching:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ s_4 & s_2 & s_3 & s_1 & s_5 \end{pmatrix}$$

giving i_5 her **third** preference. All other students are assigned to their first or second choices. When we study the preference profiles, we see that each school shows up in some preference profile by level 3.

Consider next the following preference profile:

$$i_1 : s_1 \succ s_4 \succ s_3 \succ s_2$$
$$i_2 : s_1 \succ s_4 \succ s_3 \succ s_2$$
$$i_3 : s_1 \succ s_4 \succ s_3 \succ s_2$$
$$i_4 : s_2 \succ s_3 \succ s_1 \succ s_4$$

We notice easily that all schools show up by preference level 2, but the conflict in the students' preferences means that there is no way to assign everyone in such a way that all students get their first or second choice.

We make the following definitions in order to make precise our observations above and to expand upon the notion of preference compatibility from §§2.1:

Definition 2.4. Let $\mathbf{P} = \{P_1, P_2, \cdots, P_m\}$ be a set of complete preference vectors of the m students in I. Let:

$$S_a = \{s \in S : \varphi_i(s) = a \text{ for some } i \in I\},$$

and let n be the smallest integer with $\cup_{a=1}^{n} S_a = S$. We say that $\mathbf{P}$ has **rank** n.

Definition 2.5. Let $\mathbf{P} = \{P_1, P_2, \cdots, P_m\}$ be a rank-n set of complete preference vectors of the m students in I. We say that a matching $M : I \to I \times S$ has **rank** n and is **rank-compatible** if for each $i \in I$, $\varphi_i(M_i) \leq n$. If such a matching exists we say that $\mathbf{P}$ is **rank-n-compatible** or simply **rank-compatible**.

We can now see that the first preference profile given above is rank-3, and rank-3 compatible, and the associated minimum index matching proposed is rank-compatible. The second profile is rank-2 but it is not rank-2 compatible, and hence no rank-2 matching exists.

Here, finally, is our second new criterion for the desirability of matching mechanisms:

Definition 2.6. Let $\mathfrak{P}$ be the set of all possible preference profiles for the m students in I and let $\mathfrak{M}$ be the set of all possible matchings between I and S. We say that a matching mechanism $\mathcal{M} : \mathfrak{P} \to \mathfrak{M}$ is **rank-compatible** if $\mathcal{M}(\mathbf{P})$ is rank-compatible for all rank-compatible $\mathbf{P} \in \mathfrak{P}$.

This new criterion is incompatible with stability. In fact we can prove:

Theorem 2.7. *No stable mechanism is rank-compatible for* $|I| \geq 4$, $|S| \geq 4$.

Proof. It suffices to find a rank-compatible preference profile $\mathbf{P}$ and an accompanying priority structure $\mathbf{\Pi}$ such that the pair $(\mathbf{P}, \mathbf{\Pi})$ admits no rank-compatible stable matchings. To this end consider:

$$\begin{array}{ll} i_1 : s_1 \succ s_4 \succ s_2 \succ s_3 & \qquad s_1 : i_1 \succ i_2 \succ i_3 \succ i_4 \\ i_2 : s_2 \succ s_4 \succ s_3 \succ s_1 & \qquad s_2 : i_2 \succ i_3 \succ i_4 \succ i_1 \\ i_3 : s_3 \succ s_4 \succ s_2 \succ s_1 & \qquad s_3 : i_3 \succ i_4 \succ i_1 \succ i_2 \\ i_4 : s_1 \succ s_2 \succ s_3 \succ s_4 & \qquad s_4 : i_4 \succ i_1 \succ i_2 \succ i_3 \end{array}$$

Here, the only stable matching is:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_1 & s_2 & s_3 & s_4 \end{pmatrix},$$

so any stable mechanism must output this matching. Note, however, that $S_1 \cup S_2 = S$ and so $\mathbf{P}$ has rank 2. Furthermore it is rank-2-compatible; the following is a rank-compatible matching for it:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_4 & s_2 & s_3 & s_1 \end{pmatrix}.$$

But in the stable matching i_4 receives her fourth school, and thus no stable mechanism will be rank-compatible. $\square$

We can now show that mechanisms studies earlier are not rank-compatible:

Proposition 2.8. *Most mechanisms studied earlier are NOT rank compatible. In particular,*

(1) *The Student-Optimal Stable Matching (SOSM) Mechanism* [1] *is not rank-compatible.*
(2) *The Efficiency Adjusted Deferred Acceptance Mechanism (EADAM)* [16] *is not rank-compatible.*
(3) *The Top Trading Cycles (TTC) Mechanism* [1] *is not rank-compatible.*

Proof. SOSM is stable so Part (1) is a direct corollary of the previous theorem. Part (2) follows from the fact that there are no interruptors for the $(\mathbf{P}, \mathbf{\Pi})$ pair given in the proof of Theorem 2.7 and hence EADAM reduces to SOSM in this case. Similarly, we prove Part (3) by noting that, in the same example, the outcome of the TTC mechanism is coincidentally the same as the outcome of SOSM. $\square$

2.3. **Index and Rank.** The two new notions we have introduced, preference index and rank-compatibility, are indeed closely related. We now focus on this relationship.

We first need to introduce some new notation. Let $\mathbf{P}$ be a preference profile and define $S_1, \ldots, S_n$ as in Definition 2.4. Let $S_0 = \emptyset$. For each a, define

$$S_a^C = \bigcup_{k=0}^{a} S_k - \bigcup_{k=0}^{a-1} S_k.$$

Thus, S_a^C is the set of all schools which are the ath choice for some student but not the bth choice for any student for $b < a$. Note that for us, S_a^C represents the marginal contribution of S_a to the compatibility of a preference profile. Clearly, for some a, S_a^C may be empty.

Example 2.9. Consider a previous preference profile $\mathbf{P}$ given by

$$\begin{aligned} i_1 &: s_1 \succ s_2 \succ s_4 \succ s_3 \\ i_2 &: s_2 \succ s_3 \succ s_4 \succ s_1 \\ i_3 &: s_3 \succ s_1 \succ s_4 \succ s_2 \\ i_4 &: s_1 \succ s_2 \succ s_3 \succ s_4 \end{aligned}$$

Note that $\mathbf{P}$ is rank-3-compatible. Here $S_1^C = \{s_1, s_2, s_3\}$. Furthermore, since consideration of S_2 yields no additional schools, we see that $S_2^C = \emptyset$. Finally since consideration of S_3 introduces s_4, we see that $S_3^C = \{s_4\}$.

Now we can describe the relationship between the two concepts introduced:[6]

Theorem 2.10. *Let* $\boldsymbol{P}$ *be a rank-n-compatible preference profile for* $|I| = |S| = m$ *and for each* a *define* S_a, S_a^C *as above. Then for all matchings* $M \in \mathfrak{M}$*, we have a natural lower bound on the preference index:*

$$\sum_{a=1}^{n} (a-1) \cdot |S_a^C| \leq \mu(M).$$

If furthermore M *is rank-compatible, we also have an upper bound:*

$$\mu(M) \leq mn - m.$$

[6]Note that in the following theorem as well as in a majority of our examples, we assume that $|I| = |S|$. This may not always be the case, even if we take S to be made up of individual seats in each school rather than the schools themselves. This will not cause any substantial problems for our work here however; we will assume that there will always be a seat available for each student and express this as $|S| \geq |I|$.

Proof. Since S_a^C represents the schools introduced when considering the a-th set of preferences, given a preference profile $\mathbf{P}$, the best case feasible matching occurs by awarding $|S_1^C|$ students their first choice school, $|S_2^C|$ students their second choice school, and so on. Since awarding a student their a-th choice adds $a-1$ to the preference index, we have our result.

Now we assume that $M \in \mathfrak{M}$ is a rank-compatible matching. Since $1 \leq |S_1^C|$ and $\sum_{a=1}^{n} |S_a^C| = |S| = m$, the worst possible case is if every student receives their nth choice. Hence

$$\mu(M) \leq (n-1)(m) = mn - m$$

as desired. $\square$

Now we shall revisit our previous example.

Example 2.11. Consider once again the preference profile $\mathbf{P}$ given by

$$\begin{aligned} i_1 &: s_1 \succ s_2 \succ s_4 \succ s_3 \\ i_2 &: s_2 \succ s_3 \succ s_4 \succ s_1 \\ i_3 &: s_3 \succ s_1 \succ s_4 \succ s_2 \\ i_4 &: s_1 \succ s_2 \succ s_3 \succ s_4 \end{aligned}$$

We know that $\mathbf{P}$ is rank-3-compatible. Our theorem demonstrates that any matching M needs to satisfy

$$(1-1)(3) + (2-1)(0) + (3-1)(1) = 2 \leq \mu(M),$$

and furthermore when M is rank-compatible that

$$2 \leq \mu(M) \leq 8.$$

These bounds allow us to establish the limitations of any feasible matching given a preference profile; reaffirming the well-understood notion that preferences are best honored when heterogenous. Furthermore they similarly justify the intuition that when desirable education is scarce and students desire similar schools, they will often be disappointed.

3. A cost-minimizing school choice mechanism

In §2 we defined two new notions, the preference index and rank compatibility. We then showed that previously proposed school choice mechanisms do not perform optimally with respect to these. In this section we describe an alternative mechanism geared specifically toward these two notions. Our mechanism happens to have several other desirable properties which we elaborate on in §§3.3.

Our new mechanism is built upon a combinatorial optimization algorithm known as the Hungarian algorithm. The Hungarian algorithm is traditionally used to find the minimum cost matching in various **min-cost max-flow problems** such as assigning individuals to tasks or determining minimum cost networks in travel [17], [18]. We note that the algorithm can be processed in polynomial time [21], hence the mechanism itself can effectively be implemented via a computer program.

As the purpose of the Hungarian algorithm is to find the minimum cost matching, the first step in adapting the algorithm to the school choice problem would be to define the cost of any particular matching. But we already have a natural candidate. Indeed the preference reverence index defined in §§2.1 proves to be a good indicator of the cost of a given matching by measuring the cost in terms of the number and extent of preference violations. Thus the resulting mechanism naturally minimizes the preference reverence index. Later in the section we introduce a second, alternative concept of cost (see §§3.4). Running our mechanism with this choice of cost leads to optimal outcomes with respect to our second measure of desirability from the previous section, rank-compatibility.

In the rest of this section we focus on various aspects of using the Hungarian algorithm in the school choice problem. We first describe the standard Hungarian algorithm for assignment problems (§§3.1). We then explain how we adapt it to our context with cost determined by the preference reverence index (§§3.2). Next we study some properties of this "Hungarian" school choice mechanism (§§3.3). We introduce our second notion of cost and investigate a rank-compatible modification of our mechanism in §§3.4. We discuss some implementation issues in §§3.5.

3.1. **Description.** Let I and S be a set of students and schools, respectively, and assume that a student preference profile $\mathbf{P}$ is given. Suppose the students are preference incompatible. Since the space $\mathfrak{M}$ of all matchings is finite, $\mu(\mathfrak{M})$ is finite and therefore there exists some $M \in \mathfrak{M}$ such that $\mu(M) \leq \mu(M')$ for all $M' \in \mathfrak{M}$. We will now find such a minimal index matching.

Let $[A] = (a_{jk})$ be the matrix such that $a_{jk} = \varphi_{i_j}(s_k)$. We are essentially embedding student preferences in an $n \times m$ matrix. This is necessary in order to run the Hungarian algorithm. For now we will assume that $n = m$, i.e. there is an equal number of students and schools and each school has a capacity of one.

For example if we are given the following preference profile of three students for three schools:

$$\begin{aligned} i_1 &: s_1 \succ s_2 \succ s_3 \\ i_2 &: s_3 \succ s_2 \succ s_1 \\ i_3 &: s_2 \succ s_3 \succ s_1 \end{aligned}$$

then we could embed these preferences in a matrix as:

	s_1	s_2	s_3
i_1	1	2	3
i_2	3	2	1
i_3	3	1	2

We want to find one element in exactly one row and one column such that the sum of the entries is minimal. The Hungarian algorithm will run as follows:

(1) Subtract 1 from each row.
(2) For each column i, let ϵ_i denote the least element in that column. For each column i, subtract ϵ_i from that column.
(3) If there is a 0 entry in each column and row, the assignment is complete and we are done. If not, let V denote the set of columns and rows that have a zero entry in them.
(4) Let $\Delta = \min_{j \notin V, k \notin V} a_{jk}$. Let $[A'] = (c_{jk})$ be the matrix achieved by letting:

$$c_{jk} = \left\{ \begin{array}{ll} a_{jk} - \Delta & j \notin V \text{ and } k \notin V \\ a_{jk} & j \in V \text{ or } k \in V \\ a_{jk} + \Delta & j \in V \text{ and } k \in V \end{array} \right\}.$$

(5) Repeat steps $1 - 4$ until finished.

Here is the outcome of the Hungarian algorithm for the preference profile used in the proof of Theorem 2.7:

	s_1	s_2	s_3
i_1	$\boxed{1}$	2	3
i_2	3	2	$\boxed{1}$
i_3	3	$\boxed{1}$	2

Since the matching algorithm produces the "least cost" matching [21], and our cost is represented by the preference index, we see that the resultant matching has the smallest preference index with respect to each student's preferences. Hence we have (also see Theorem 3.2):

Proposition 3.1. *Given a set of preferences, let M be the matching produced by the Hungarian algorithm and let M' be any other matching. Then $\mu(M) \leq \mu(M')$.*

3.2. **A "Hungarian" school choice mechanism.** In adapting the Hungarian algorithm to the school choice problem, we must make three key modifications. The construction of the algorithm requires as input an $n \times n$ matrix of non-negative numbers, and it selects as output a unique entry in each row and each column. Therefore, we must modify the algorithm to accommodate school capacities, unequal numbers of seats and students, as well as preferences containing different numbers of ranked schools. In the following we will call our modified mechanism the *Index-Based Hungarian Mechanism* and denote it by $\mathcal{HM}_i$, where i emphasizes our focus on the preference index.

Assume columns represent schools and rows represent students in our matrix. To express school capacities, we simply add an extra column for each available seat at a school and enter the same preferences for that column.[7] Thus, each column would represent a seat at a school, rather than an entire school. Then, if there are an unequal number of total available seats and students (i.e. an unequal number of rows and columns), we add any necessary dummy rows or dummy columns, which represent nonexistent students or schools. Thus, if a dummy row student were assigned to an actual school, this would signify an open seat at that school, whereas if an actual student were assigned a dummy column school this would signify that that student has remained unassigned by the mechanism.

The third modification addresses the problem of families submitting incomplete preference profiles. It is not immediately obvious why a district should request that students submit lists of the same length. Furthermore, even if a school district requires students to rank a specific number of schools, there will undoubtedly be students who do not list the required number. Regardless, in order to run the Hungarian algorithm, it is necessary to devise a way of completing student preferences such that each student preference list assigns a rank to each school or seat.

A potential solution is to use dummy variables to complete any missing entries in the matrix. However, this method may invite students to strategize. Even without complete information, students might be motivated to strategize by only submitting their first choice school, thereby weighting this choice with dummy variables so that the algorithm is more likely to select it.

We instead propose an alternative way to complete student preferences that preserves the strategyproofness of the Index-Based Hungarian Mechanism. Our method also incorporates school priorities and as such may be appealing for districts where these should be emphasized. More specifically, we propose to complete "missing" student preferences by considering the schools at which a student has the highest priority. Here, we refer not to the priority numbers that are determined by lottery within each priority grouping, but to the general priority grouping. If a student possesses equally high priority groupings among several schools, ties will be broken by the students proximity to the school. Similarly, if the student has no remaining priorities at any school and there still remain missing preferences, then their preferences are completed in order of nearest schools. To determine when a preference profile is "complete," school districts can choose to impose a cutoff on the number of ranked schools that will be considered, or to simply take the maximum number of schools a student has ranked and complete all other student preference lists to that number.

The main rationale behind completing preferences in this manner is that it is preferable to assign students, who would have otherwise remained unassigned, to schools based on their priorities

[7]Thus the matrix could have some repeated entries. In fact students could even submit non-strict rankings. In this manner the Hungarian Mechanisms introduced in this paper allow students to display indifferences between various schools with no penalty. We will say a bit more on this in Section 4.

rather than to assign them administratively.[8] Students who are assigned administratively are assigned to schools with available seats after the mechanism has run for all other students. In other words, students who submit fewer preferences than others are only offered the "leftovers." Such a policy might inadvertently punish those students whose families are unfamiliar with their district's educational system. We instead take the view that, if a student is not able to secure a seat at a school she listed, her priorities at other schools should entitle her an equal opportunity to compete for a spot at those schools. Thus, the only scenario in which a student would remain unassigned by the Index-Based Hungarian Mechanism is if there were more students than total available seats, which should not occur in the public school setting (cf. Footnote 6).

Alternatively we can fill out the remainder of a student preference profile with an equal ranking for all unranked schools. More specifically if a student's preference profile contains only r ranks, then we assign the rank $r+1$ to all the remaining schools. This incentivizes the completion of preference lists, since otherwise all remaining schools will be treated equally. For instance, if a family puts only their first choice, all other choices will be considered "second"; therefore they may get a school which they consider terrible at low cost as measured by the mechanism. Thus it would behoove them to fill out as many schools as possible if they had a genuine preference for one over another. This method does not take into account priorities, but it does deflect strategizing. We offer it here as a simpler alternative.

3.3. **Properties.** Since the Index-Based Hungarian Mechanism ($\mathcal{HM}_i$) selects the "least cost" matching, when we use preferences as "costs" we have the following result:

Theorem 3.2. *Given a preference profile $\boldsymbol{P}$, for any mechanism $\mathcal{M}$*

$$\mu(\mathcal{HM}_i(\mathbf{P})) \leq \mu(\mathcal{M}(\mathbf{P})).$$

The Index-Based Hungarian Mechanism meets our utilitarian standard of student optimal matchings. Furthermore, it satisfies the standard notion of efficiency:

Corollary. *Given a set of preferences $\boldsymbol{P}$, let $\mathcal{HM}_i(\boldsymbol{P})$ be the matching produced by the Index-Based Hungarian Mechanism. Then $\mathcal{HM}_i(\boldsymbol{P})$ is (Pareto) efficient.*

Proof. It follows from our construction of μ that a matching M Pareto dominates a matching M' only if $\mu(M) < \mu(M')$. If $\mathcal{HM}_i(\mathbf{P})$ is not (Pareto) efficient, then there exists some matching M that (Pareto) dominates $\mathcal{HM}_i(\mathbf{P})$. But then

$$\mu(M) < \mu(\mathcal{HM}_i(\mathbf{P})),$$

and hence there exists a matching with a preference index less than $\mathcal{HM}_i(\mathbf{P})$. Since $\mathcal{HM}_i(\mathbf{P})$ has the minimum preference index among all possible matchings, this is a contradiction. □

We will next examine the performance of the $\mathcal{HM}_i$ with respect to strategic action. We first show that, under complete information, it is not strategyproof:

Proposition 3.3. *The $\mathcal{HM}_i$, under complete information, is not always strategyproof.*

Proof. We show this by producing an example of a preference profile with which it is possible for a student to strategize in order to improve her outcome: Assume that four students submit the

[8] One can also argue that this incorporates the Rawlsian principle of equal opportunity, by not only actively preventing discrimination but also even possibly giving the disadvantaged an extra boost. cf. [23, p.63]

following *truthful* preferences for four schools with one seat each:

$$\begin{aligned} i_1 &: s_1 \succ s_2 \succ s_3 \succ s_4 \\ i_2 &: s_3 \succ s_4 \succ s_2 \succ s_1 \\ i_3 &: s_2 \succ s_3 \succ s_1 \succ s_4 \\ i_4 &: s_1 \succ s_3 \succ s_4 \succ s_2 \end{aligned}$$

Here is the unique preference index minimizing solution output by $\mathcal{HM}_i$:

	s_1	s_2	s_3	s_4
i_1	$\boxed{1}$	2	3	4
i_2	4	3	$\boxed{1}$	2
i_3	3	$\boxed{1}$	2	4
i_4	1	4	2	$\boxed{3}$

We can also express the matching as:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_1 & s_3 & s_2 & s_4 \end{pmatrix}.$$

Here, notice that everyone has received his or her first choice except for i_4. However, there does exist a way in which i_4 could manipulate her preference list to receive a preferred school. Suppose i_4 strategizes by listing her second choice school as her first choice school (and listing her first choice school as her second choice school). The preference profile becomes:

$$\begin{aligned} i_1 &: s_1 \succ s_2 \succ s_3 \succ s_4 \\ i_2 &: s_3 \succ s_4 \succ s_2 \succ s_1 \\ i_3 &: s_2 \succ s_3 \succ s_1 \succ s_4 \\ i_4 &: s_3 \succ s_1 \succ s_4 \succ s_2 \end{aligned}$$

Here is the unique preference index minimizing solution output by $\mathcal{HM}_i$ in this case:

	s_1	s_2	s_3	s_4
i_1	$\boxed{1}$	2	3	4
i_2	4	3	1	$\boxed{2}$
i_3	3	$\boxed{1}$	2	4
i_4	2	4	$\boxed{1}$	3

Thus, the matching under the falsified preferences is:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_1 & s_4 & s_2 & s_3 \end{pmatrix}.$$

Notice that i_4 has now received s_3 whereas before, under the truthful preferences, i_4 received s_4. If we look at her truthful preferences, we see i_4 did in fact prefer s_3 to s_4. Thus, i_4 has successfully strategized by manipulating her preference list to receive a preferred school. □

Thus under complete information, the Index-Based Hungarian Mechanism is not always immune to strategic action. However, we are more interested in its performance under incomplete information. In this latter situation, the question is not whether it is *possible* for a student to receive a preferred school by lying, but whether any student would have a *rational* motive to falsify preferences. In the macro school choice environment, players are not privy to everyone else's preferences. Knowing the preferences of a large number of students is not enough; a single unknown row (a single unknown student) can change the final outcome of $\mathcal{HM}_i$. Furthermore, as listing a desired school higher increases one's chance of receiving that school and does not penalize or "waste" his second choice school (cf. the Boston Mechanism [4], [1]) the possibility of parents "settling" or strategizing as

such is minimal. For all practical purposes, the Index-Based Hungarian Mechanism with incomplete information does not incentivize lying, or in other words, there does not exist any valid rationale for students which might incentivize lying when this procedure is used.

One might suspect that the way we complete preferences by using priorities (cf. §§3.2) eliminates strategyproofness under incomplete information. Indeed, a student who did not submit the requested number of schools might receive a school she did not ask for. In such cases, had we not completed the student's preferences, she would have remained unassigned by the mechanism (and therefore, would have had to be administratively assigned.) Thus, it is possible that the school she received after we completed her preferences was preferred to the school she would have received had she been administratively assigned. In other words, it is *possible* for a student to be made better off (relative to what their administrative assignment would have been) if we complete their preferences using priorities and run $\mathcal{HM}_i$.

While it is possible that this can happen, it is never part of a strategy. That is, no student ever has an incentive to strategize by omitting preferences since a student can never receive a preferred assignment by omitting preferences relative to submitting complete preferences. Omitting assignments can only hurt students (because we might complete their preferences in a way which they did not prefer), and can never help them (at best, if we are lucky, we complete their preferences as they would have done themselves).

Thus we have shown that the $\mathcal{HM}_i$ is Pareto efficient, minimizes the preference index and is strategyproof with incomplete information. Despite its success with respect to criteria new and old however, we also recognize potential limitations of this approach. We summarize these limitations in the following.

Proposition 3.4. (i) *The Index-Based Hungarian Mechanism is not stable.*
(ii) *The Index-Based Hungarian Mechanism is not rank-compatible.*

Proof. Consider the following preference profile $\mathbf{P}$:

$$\begin{aligned}
i_1 &: \; s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5 \\
i_2 &: \; s_2 \succ s_3 \succ s_4 \succ s_5 \succ s_1 \\
i_3 &: \; s_3 \succ s_4 \succ s_5 \succ s_1 \succ s_2 \\
i_4 &: \; s_4 \succ s_5 \succ s_1 \succ s_2 \succ s_3 \\
i_5 &: \; s_1 \succ s_2 \succ s_5 \succ s_3 \succ s_4
\end{aligned}$$

and assume it is accompanied with the priority structure $\mathbf{\Pi}$:

$$\begin{aligned}
s_1 &: \; i_1 \succ i_5 \succ i_2 \succ i_3 \succ i_4 \\
s_2 &: \; i_5 \succ i_3 \succ i_2 \succ i_1 \succ i_4 \\
s_3 &: \; i_3 \succ i_1 \succ i_2 \succ i_4 \succ i_5 \\
s_4 &: \; i_4 \succ i_3 \succ i_2 \succ i_1 \succ i_5 \\
s_5 &: \; i_1 \succ i_5 \succ i_2 \succ i_3 \succ i_4
\end{aligned}$$

The preference profile $\mathbf{P}$ has rank 2, and moreover is rank-2-compatible as it is possible to assign everyone to their first or second choice schools. However, the Hungarian algorithm selects

$$\mathcal{HM}_i(\mathbf{P}) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ s_1 & s_2 & s_3 & s_4 & s_5 \end{pmatrix},$$

where i_5 receives her third choice. Since our profile was rank-2 compatible, we see that the Index-Based Hungarian Mechanism is not rank-compatible. Furthermore, with the given priority structure $\mathbf{\Pi}$, this outcome is unstable as i_2 is violating the priority of i_5 for s_2. □

We observed earlier that the SOSM outcome is the minimum preference index stable matching (Lemma 2.3). Thus in our framework, we can identify the SOSM outcome as the solution to the

school choice problem that minimizes the preference index subject to the constraint of stability. In this context, the $\mathcal{HM}_i$ outcome is the solution to the school choice problem that minimizes the preference index with no constraint. Then, one might ask if making everyone best off relative to each other is "worth" potentially high numbers of priority violations. The cost of inefficiency imposed by stability has been studied extensively in the literature, but the cost of instability imposed by efficiency has been modeled and studied much less often (but also see [28] for a discussion of a similar trade-off between efficiency and equity).

We can offer a remedy to those who would prefer fewer priority violations to more and who are willing to sacrifice some efficiency. With respect to the preference index, we showed that the $\mathcal{HM}_i$ outcome is optimal. However, determining the "next best matching" with respect to the preference index is paramount for a district wishing to minimize priority violations. We can do so by listing assignments in order of increasing preference index using a method of [22]. Because the SOSM outcome is the lowest preference index stable matching, it will be the first stable matching found in this list.

Alternatively, one might start with the $\mathcal{HM}_i$ output and impose stable cycle improvements a la [9]. Each improvement would correspond to a decrease in priority violations, and thus produce a more stable matching. Counting the number of priority violations, policy makers can determine what a particular student's priority is worth in terms of others' preferences.

An important observation to make is that in basing our mechanism on the preference index, we are merely attempting to refine a (rather coarse) partial order. This is not to say that the $\mathcal{HM}_i$ output will Pareto dominate matchings obtained via all previous mechanisms. Certainly the $\mathcal{HM}_i$ will not Pareto dominate TTC, given that TTC is Pareto efficient. Similarly, when the SOSM outcome is Pareto efficient, $\mathcal{HM}_i$ will not Pareto dominate it.

3.4. **A Rank-Compatible Hungarian Mechanism.** In the previous sections, we described a mechanism that finds minimum preference index matchings by embedding student preferences in a matrix and running the Hungarian algorithm. Recall that the value for the (i, j)th entry in the matrix was simply $\varphi_i(s_j)$, student i's ranking of school s_j. While this mechanism is optimal with respect to the goal of minimizing the preference index, it does not satisfy our second criterion of rank-compatibility. In this subsection we will again use the Hungarian algorithm, this time to find rank-compatible matchings by modifying how we embed student preferences in a matrix.

Recall that a mechanism is rank-compatible if it always produces matchings in which the greatest overall ranking number of an assignment received could not be reduced in any other matching. In other words, a mechanism is rank-compatible if it always gives the "worst-off" student(s) as preferred of an assignment as possible. To adapt the Hungarian algorithm to satisfy this criterion, we weight preferences in our matrix such that a rank-compatible matching is cost-minimizing. We make it as "costly" to assign *one* student their j^{th} ranked school as it is to assign *all* students their $j-1$ ranked schools. To this end, we define the function that determines each entry in the matrix as:

$$\zeta_i = n^{\varphi_i(s)},$$

where n is the total number of students. This ensures that:

$$n^{\varphi_i(s)} = n(n^{\varphi_i(s)-1}).$$

Apart from this new method of weighting preferences in our matrix, our procedure for running the Hungarian algorithm remains exactly the same. In particular, all of our previously described methods of completing preference lists using priorities and proximity, representing school capacities, using dummy variables to ensure a square matrix, etc, still apply. We will call this new mechanism the Rank-Based Hungarian Mechanism, and denote it by $\mathcal{HM}_r$.

Theorem 3.5. *$\mathcal{HM}_r$ is rank-compatible.*

Proof. For a particular student, let $\varphi_i(s) = j$, where j is student i's ranking number for a particular school s. Assume the $\mathcal{HM}_r$ assigned this student their j^{th} ranked school when it was possible to assign *all* students to any combination of schools they preferred to their own individual j^{th} ranked schools. Recall that we defined each entry in our matrix as $n^{\varphi_i(s)}$ where n was the total number of students and $\varphi_i(s)$ was each student i's ranking number for a particular school s. This implies that any single entry for a student's j^{th} ranked school will be greater than or equal to the sum of each students entry for a school they preferred to their own j^{th} ranked schools. More precisely: Fix some $j, n \in \mathbb{N}$. Assume $0 < \varphi_i(s) < j$ where $\varphi_i(s) \in \mathbb{N}$. Then it follows that:

$$n^j \geq \sum_{i=1}^{n} n^{\varphi_i(s)}.$$

Thus, if the Rank-Based Hungarian Mechanism were to assign one student their j^{th} ranked school when it was possible to assign all students to more preferred (ranked less than j) schools, this would contradict the fact that the Hungarian algorithm matching minimizes the sum of the selected entries of the matrix. Therefore, the $\mathcal{HM}_r$ is rank-compatible. $\square$

In defining the value for each entry in our matrix as $\zeta_i = n^{\varphi_i(s)}$, we inadvertently define a new index which we will call ω. Define $\omega : \mathfrak{M} \to \mathbb{N}$ by

$$\omega(M) = \sum_{i \in I} \zeta_i(M_i).$$

We will not be as concerned with ω itself as we were with the preference index μ. While we designed the Index-Based Hungarian Mechanism to minimize μ, we did not design the $\mathcal{HM}_r$ to necessarily minimize ω. This is because, in spite of being constructed in order to produce rank-compatible matchings, ω sometimes fails to distinguish between rank-compatible matchings. Furthermore, while neither μ nor ω is intended as a literal measurement of welfare; μ attempts to best honor ordinal preferences by making the safer assumption of uniform utility gaps between preferences, whereas ω makes the radical assumption of exponential utility gaps. Thus, we note that we introduce ω for the sole purposes of aiding us in our discussion of the properties of the $\mathcal{HM}_r$.

Lemma 3.6. *Let M, $M' : I \to I \times S$ be two matchings. If M (Pareto) dominates M', then $\omega(M) < \omega(M')$.*

Proof. Since M dominates M', there exists at least one student i who is better off under M than M'. In other words $\varphi_i(M_i) < \varphi_i(M'_i)$. Since for all $j \in I \backslash \{i\}$, we have $\varphi_j(M_j) \leq \varphi_j(M'_j)$, this implies $\zeta_j(M_j) \leq \zeta_j(M'_j)$, which yields $\omega(M) < \omega(M')$. $\square$

Since $\mathcal{HM}_r$ uses the Hungarian algorithm to minimize the sum of the selected entries, the index that is associated with the sum of those entries (which we named ω) will of course be minimized. Thus, we have:

Theorem 3.7. *Given a preference profile $\boldsymbol{P}$, for any mechanism $\mathcal{M}$*

$$\omega(\mathcal{HM}_r(\mathbf{P})) \leq \omega(\mathcal{M}(\mathbf{P})).$$

Corollary. *Given a set of preferences $\boldsymbol{P}$, let $\mathcal{HM}_r(\boldsymbol{P})$ be the matching produced by the Rank-Based Hungarian Mechanism. Then $\mathcal{HM}_r(\boldsymbol{P})$ is (Pareto) efficient.*

Proof. It follows from our construction of ω that a matching M Pareto dominates a matching M' only if $\omega(M) < \omega(M')$. If $\mathcal{HM}_r(\mathbf{P})$ is not (Pareto) efficient, then there exists some matching M that (Pareto) dominates $\mathcal{HM}_r(\mathbf{P})$. But then

$$\omega(M) < \omega(\mathcal{HM}_r(\mathbf{P})),$$

and hence there exists a matching with a ω index less than $\mathcal{HM}_r(\mathbf{P})$. Since $\mathcal{HM}_r(\mathbf{P})$ should have the minimum ω index this is a contradiction. $\square$

With regard to the strategyproofness of the $\mathcal{HM}_r$, we refer the reader to our discussion in §§3.3 of the Index-Based Hungarian Mechanism under strategic action since an identical argument holds here: It is conceivable that, under complete information, a student could successfully manipulate preferences to receive a preferred school. However, under incomplete information, it is in all students' best interest to state their preferences truthfully. Thus, for all practical purposes, the $\mathcal{HM}_r$ is strategyproof.

Lastly, we note the limitations of $\mathcal{HM}_r$.

Proposition 3.8. (i) *The $\mathcal{HM}_r$ is not stable.*
(ii) *The $\mathcal{HM}_r$ is not preference-index (μ) minimizing.*

Proof. Using the same preference list and priority structure given in Proposition 3.4, we see that the Rank-Based Hungarian Mechanism finds:

$$\mathcal{HM}_r(\mathbf{P}) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ s_1 & s_3 & s_4 & s_5 & s_2 \end{pmatrix}.$$

Here, i_2 is violating the priority of i_3 for s_3, and this suffices for (i). For (ii), we note that the above matching has preference index $\mu = 4$ while the Index-Based Hungarian Mechanism has $\mu = 2$. $\square$

3.5. **An implementation issue: multiple minima.** As the example before Lemma 2.3 in §§2.1 shows, the ordering induced by the preference index is not strict. Indeed a given preference profile might have multiple minimum preference index solutions. For example, consider the following preference profile:

$$\begin{aligned} i_1 &: s_1 \succ s_2 \succ s_3 \succ s_4 \\ i_2 &: s_4 \succ s_2 \succ s_1 \succ s_3 \\ i_3 &: s_3 \succ s_1 \succ s_4 \succ s_2 \\ i_4 &: s_3 \succ s_4 \succ s_2 \succ s_1 \end{aligned}$$

Here, the Index-Based Hungarian Mechanism only produces Matching 1, while there are in fact three minimum preference index matchings:

$$\text{Matching \#1} \quad (\mu = 2): \qquad \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_1 & s_2 & s_3 & s_4 \end{pmatrix}$$

$$\text{Matching \#2} \quad (\mu = 2): \qquad \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_2 & s_4 & s_1 & s_3 \end{pmatrix}$$

$$\text{Matching \#3} \quad (\mu = 2): \qquad \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_1 & s_4 & s_3 & s_2 \end{pmatrix}$$

The underlying theoretical problem of finding all possible minimum cost assignments by the Hungarian algorithm was addressed and answered in [12]. Thus it is possible to find all minimum preference index solutions. This in turn raises a new question: How might one choose among all these minimum preference index solutions? We propose two possible approaches to deal with this issue.

(1) If one intends to promote fairness by narrowing the discrepancies between the rankings of student assignments, then the matching with the minimum variance across individual student preference indices should be chosen. This promotes fairness from a Rawlsian perspective by making the worst off student better off.
(2) If one intends to choose "the most stable" matching, then the matching with the fewest instances of priority violations should be chosen. Here, we look at the number of students who had their priorities violated by other students.[9]

Of course, it is also possible that one could use both of these criteria in succession.

This incidentally helps us resolve a possible concern about the Hungarian algorithm: its visible dependency on the order of the rows and the columns of the input matrix. Especially cases where there may be multiple minimal index solutions, the order in which students or schools are listed may indeed affect the outcome, and the output matching may be different in different cases (though any two outcomes in such a scenario will always have the same index). However if we modify our mechanism to look instead for all possible minimum cost matchings, this will no longer create a problem. Thus, the order of the rows or columns ultimately doesn't matter for our purposes because: (1) If there is a unique cost minimizing solution, the order does not affect the outcome at all; and (2) if there are multiple cost minimizing solutions, we can still find all of them using our mechanism, with adaptations a la [12].

The situation is somewhat different in the case of the Rank-Based Hungarian Mechanism. If we were to use [12] to find all minimum ω index matchings, we would not necessarily find all rank-compatible matchings. This is because there might exist two rank-compatible matchings for a given preference profile that have different ω costs. For example, consider the following preference profile:

$$i_1 : s_3 \succ s_2 \succ s_1$$
$$i_2 : s_2 \succ s_3 \succ s_1$$
$$i_3 : s_2 \succ s_1 \succ s_3$$

There are two matchings which both have the minimum rank-2 but have different ω costs:

$$\text{Matching 1 (Rank-2, } \omega = 15, \mu = 1) : \qquad \begin{pmatrix} i_1 & i_2 & i_3 \\ s_3 & s_2 & s_1 \end{pmatrix}$$

$$\text{Matching 2 (Rank-2, } \omega = 27, \mu = 3) : \qquad \begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_3 & s_1 \end{pmatrix}$$

Here, Matching 1 is the minimum ω index matching that would have been found by the Rank-Based Hungarian Mechanism, whereas Matching 2 is another matching with the same minimal rank. However, notice that Matching 1 Pareto dominates Matching 2.

In fact, all minimum ω index matchings are not only rank-compatible, but are also Pareto efficient (see Corollary for Theorem 3.7). Thus, we see that not all rank-compatible matchings are necessarily Pareto efficient, while rank-compatible matchings with minimal ω index are necessarily Pareto efficient. We can again use [12] to find all minimum ω index (Pareto efficient and rank compatible) matchings, and employ the two aforementioned tie-breaking criteria. Furthermore, if one wanted to find all possible rank-compatible matchings, we can still propose a heuristic theoretical method to attack this problem: *List all possible matchings in order of increasing ω cost. Every matching that is listed before the first matching that changes rank will be rank-compatible.*

[9] This is not the same as looking at the total number of priority violations since a student could have their priority violated by multiple students. We look at the former and not the latter because once a student's priority has been violated, it has been violated whether by 1 or by 100 students. Similarly, a student can pursue legal action whether their priority is violated by 1 or by 100 students. Thus, we are probably more concerned with how many student had their priority violated rather than how many total priority violations there are.

4. Conclusion

Mechanisms commonly applied to the school choice problem focus on balancing student preferences and school priorities. In order to respect both, the resulting matches must sacrifice some desirable characteristics. Since, sadly, a good public education is currently a scarce resource, there is no way to assign students to schools in such a way that all students attend top ranked schools. Much of the controversy in school choice[10] surrounds the ability of parents to send their children to their school of choice. Thus in our approach we chose to focus on student preferences. To this end, we developed two new criteria, or measures, by which to evaluate the quality of a matching that results from a school choice mechanism. Among other things, these new measures allowed us to compare matchings/mechanisms that were incomparable in terms of previously applied criteria. The mechanisms that had been studied in the past, not surprisingly, fell short on one or the other of these new measures. This is in part a result of the natural tension between stability (which is related to priority violation) and efficiency (which is related to preferences). Essentially, our approach was to start by focusing on student preferences (efficiency) at the expense of school priorities (stability).

The two new mechanisms presented were both adapted from the well known Hungarian algorithm [17] which was developed as a combinatorial solution to the assignment problem. Our modifications included a re-interpretation of assignments taking into account school capacities and required that we be allowed to "complete" submitted student preference profiles. With the introduction of this flexibility came the requirement that we determine a fair (and strategyproof) way of completing student preference profiles. In the profile completion process we sought to avoid confounding the problem of having non-participatory parents/adults costing unknowing and often powerless children a seat at the best possible school. We also had to avoid creating the possibility that parents could exploit the process (strategize) based upon a known method of completing preference profiles.

Both of our newly proposed mechanisms provide a type of "optimization" based upon the newly introduced criteria. The *Index-Based Hungarian Mechanism*, is optimal with respect to the preference index in that it minimizes preference violations as measured by the preference index. In analyzing the performance of our Index-Based Hungarian Mechanism, we also proved that it satisfies the commonly sought criteria of strategyproofness and Pareto efficiency, as well as the minimization of the preference index. A modification to the relevant cost function allowed us to devise an analogous mechanism, the Rank-Based Hungarian Mechanism, that behaved optimally with respect to rank compatibility. Both properties emphasize accommodating student preferences and thus output matchings that better honor "student choice" than previously studied "school choice" mechanisms.

It is interesting to note that it is possible to find the SOSM solution using the preference index and the Index Based Hungarian Mechanism (cf. §§3.3). In some sense, our approach starts at the "top" of the range of assignments in regards to family preferences (minimizing preference violations), and then works down (seeking to address priority violations), while the imposition of stability in earlier approaches makes priorities *de facto* of greatest import. More precisely, if we start with the $\mathcal{HM}_i$ outcome, we can then list assignments in order of increasing preference index using a method defined in [22]. Because SOSM is the lowest preference index stable matching, we know that SOSM will be the first stable matching in this list. This "top-down" framework allows one to quantify the opportunity cost of stability in terms of preference index forgone. This supplements the fact that has already been well established in the literature: that imposing stability may come at a great cost.

An obvious weakness of our new mechanisms is instability. One method we proposed to complete preference profiles incorporates school priorities, thereby potentially alleviating the instability of

[10]See for instance several recent feature-length movies on school choice: Waiting for "Superman", The Lottery, The Cartel Movie.

the matching that results from the preference focus (cf. §§3.3). Another approach would be to follow up on the notion of priority index alluded to in Footnote 5. We give a brief example here which incorporates the costs for both the students (in terms of preference index) and the schools (in terms of priority index) in any given matching by simply summing the two indices.

Consider the following scenario where three students submit preferences over three schools:

$$\begin{aligned} i_1 &: s_1 \succ s_2 \succ s_3 \\ i_2 &: s_3 \succ s_2 \succ s_1 \\ i_3 &: s_3 \succ s_1 \succ s_2 \end{aligned}$$

Then, we encode these into a matrix as follows:

	s_1	s_2	s_3
i_1	1	2	3
i_2	3	2	1
i_3	2	3	1

Assume that the school priorities are:

$$\begin{aligned} s_1 &: i_2 \succ i_1 \succ i_3 \\ s_2 &: i_1 \succ i_2 \succ i_3 \\ s_3 &: i_2 \succ i_3 \succ i_1 \end{aligned}$$

We again embed these into a matrix (keeping students as rows and schools as columns):

	s_1	s_2	s_3
i_1	2	1	3
i_2	1	2	1
i_3	3	3	2

Now, we add the two matrices entry-wise to get a "total cost" matrix:

	s_1	s_2	s_3
i_1	3	3	6
i_2	4	4	2
i_3	5	6	3

The Hungarian algorithm outputs these two matchings, each minimizing this new notion of total cost:

$$\text{Matching \#1} \quad (\mu = 1;\ \text{Total cost} = 10): \qquad \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

$$\text{Matching \#2} \quad (\mu = 2;\ \text{Total cost} = 10): \qquad \begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_3 & s_1 \end{pmatrix}$$

Note that neither outcome is stable; indeed the minimal preference index stable matching has preference index 2 and total cost 11.

In the example above the total cost equally considers priorities and preferences. For a more sophisticated approach, one could modify the matrix entries by giving different weights to preference index vs. priority index rather than simply taking their sum. The choice of these weighting factors would necessarily be context-dependent and we leave the investigation of this thread to future work. We see an overall more robust incorporation of school priories as a very interesting direction for further investigation.

Another interesting direction for future work is in the incorporation of indifferences. In particular, our Hungarian Mechanisms afford students the opportunity to express indifferences without sacrificing strategyproofness. If a student is indifferent between several schools, the ranking number for these schools is simply repeated in the matrix. Therefore, students are not able to weight their

top choice schools by listing all of their other choices as indifferent since, once their preferences are transcribed into the matrix, the entries following their top ranked school(s) would be smaller than if they had submitted only strict preferences. In fact, any dishonest representation of indifferences can only serve to harm a student's chance of receiving their preferred schools. There has been much work focusing on indifferences in school priorities (see for instance [9]), but not as much has been done on student indifferences. We believe that this is an interesting thread to follow.

Our focus on student preferences over school priorities is natural in the current climate in which the public debate over charter schools and school vouchers rages in an attempt to offer parents more control over their children's educational choices. The fact that stability often comes at a huge cost in terms of efficiency has already been discussed and repeated at length in the School Choice literature. However, there might be cases in which, if we were to allow for simply one student with a violated priority, then we could make considerable efficiency gains (in terms of lower preference index). Our method allows for viewing "degrees of stability" (e.g. "highly stable" would correspond to a "low number" of priority violations) in light of efficiency gains (as determined by the preference index). In cases where it is possible to achieve large efficiency gains while remaining highly or semi-stable, it might be advantageous to do so through one of our Hungarian Mechanisms. Our methods should also appeal to families, since their preferences are taken into account "first." As assignments shift in an attempt to minimize preference violations and/or priority violations, some balance might be reached.

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